

Large deviation asymptotics for a random variable with Lévy measure supported by $[0, 1]$

Richard Arratia Fred Kochman Sandy Zabell

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Abstract

Asymptotics for Dickman's number theoretic function $\rho(u)$, as $u \rightarrow \infty$, were given de Bruijn and Alladi, and later in sharper form by Hildebrand and Tenenbaum. The perspective in these works is that of analytic number theory. However, the function $\rho(\cdot)$ also arises as a constant multiple of a certain probability density connected with a scale invariant Poisson process, and we observe that Dickman asymptotics can be interpreted as a Gaussian local limit theorem for the sum of arrivals in a tilted Poisson process, combined with untilting.

In this paper we exploit and extend this reasoning to obtain analogous asymptotic formulas for a class of functions including, in addition to Dickman's function, the densities of random variables having Lévy measure with support contained in $[0, 1]$, subject to mild regularity assumptions.

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1 Introduction

Dickman's function ρ is a basic function in analytic number theory, see [11, Section III.5]. It satisfies

$$\rho(u) = 0 \text{ for } u < 0, \rho(u) = 1 \text{ for } 0 \leq u \leq 1, \quad (1)$$

$$u\rho(u) = \int_{u-1}^u \rho(t)dt \quad \text{for all real } u. \quad (2)$$

Write $\psi(x, y)$ for the number of positive integers $n \leq x$, all of whose prime factors p satisfy $p \leq y$. In 1930 Dickman [7] showed that for $u > 1$,

$$\rho(u) = \lim_{x \rightarrow \infty} \frac{1}{x} \psi(x, x^{1/u}).$$

Armed with (1) and (2), a suitable Fourier representation formula, and the method of steepest descent, Hildebrand and Tenenbaum [8] reprove the classic result of de Bruijn (1951) and Alladi (1982), see [11], that as $u \rightarrow \infty$,

$$\rho(u) = \sqrt{\frac{\beta'(u)}{2\pi}} e^{\gamma - u\beta + C(\beta)} \left\{ 1 + O\left(\frac{1}{u}\right) \right\}, \quad (3)$$

where γ is Euler's constant, $\beta = \beta(u)$ is defined by the formula

$$e^\beta = 1 + u\beta,$$

and

$$C(\beta) = \int_0^\beta \frac{e^t - 1}{t} dt = \int_0^1 \frac{e^{\beta t} - 1}{t} dt.$$

(Alladi had improved on the earlier result of de Bruijn.)

Now consider the scale invariant Poisson process on $(0, \infty)$, with intensity $(1/x) dx$; see [1]. Restricting to $(0, 1]$, we have a Poisson process whose arrivals may be labeled in decreasing order, with $1 \geq X_1 > X_2 > \dots > 0$, and the sum of these arrivals, $T := X_1 + X_2 + \dots$, is the random variable characterized by its moment generating function,

$$\mathbb{E} e^{\beta T} = \exp \left(\int_0^1 \frac{e^{\beta x} - 1}{x} dx \right). \quad (4)$$

Size biasing, see [4], makes it easy to see that probability density f of T in place of ρ satisfies (2). Obviously f is zero on $(-\infty, 0)$, and scale invariance

shows that f is constant on $(0, 1)$. From this it follows that f must be a constant multiple of ρ , and knowing

$$\int_0^\infty \rho(u) du = e^\gamma,$$

(see for example [11, Formulas 5.45 and 5.43]) one sees that

$$f(u) = e^{-\gamma} \rho(u). \quad (5)$$

Thus we can rewrite (3) as a statement of the asymptotic decay of the density $f(u)$ as $u \rightarrow \infty$ (note Euler's γ no longer appears):

$$f(u) = \sqrt{\frac{\beta'(u)}{2\pi}} e^{-u\beta + C(\beta)} \left\{ 1 + O\left(\frac{1}{u}\right) \right\}. \quad (6)$$

The function C appearing in these formulas is the cumulant generating function for T , and the similarity to the large deviation results of Cramér, Chernoff, and their successors (see, for example, [6]) may be evident.

In this paper, we prove results similar to (6) for a broader class, those infinitely divisible distributions whose Lévy measure is supported on $[0, 1]$, subject to additional mild regularity conditions. From the perspective of a probabilist, the novelty of this paper is the derivation of a local limit theorem, Proposition 1, for a general case other than that of classical sums of i.i.d. variables, informally Cramér β -tilts T_β of a fixed random variable T as $\beta \rightarrow \infty$. Via untilting, this leads to asymptotic formulas, given in Theorems 1 and 2, for the density $f(u)$ as $u \rightarrow \infty$, along with a matching asymptotic formula for the upper tail probability $P(T \geq u)$, for a fixed random variable T in the class Lévy $[0, 1]$, as defined in Section 1.1. Our adaptation of the arguments from [8] and [11] eliminates, at a certain point, the use of some Whittaker–Watson species of special function theory which applies only to the Dickman case, and substitutes a more robust method.

An important example (covered by our Theorem 2) showing how variants of (3) arise naturally comes from making a “minor” change in (4), simply changing the lower limit of the integral from 0 to $a \in (0, 1)$, to get $T \equiv T^{(a)}$ with distribution characterized by

$$\mathbb{E} e^{\beta T} = \exp \left(\int_a^1 \frac{e^{\beta x} - 1}{x} dx \right). \quad (7)$$

This arises in the study of random permutations, see [2, Section 4.3]. Directly, $f(1)$ governs the asymptotic probability that a random permutation of n objects has only cycles of length at least an . Scale invariance leads to $\omega(u) = f(1)$ for the case $a = 1/u$, with Buchstab's function ω governing integers free of small prime factors; see [11, Section III.6]. Scale invariance also leads, for fixed $a \in (0, 1)$, to $f(u)$ governing the probability that a random permutation of n objects has only cycles with lengths in $(\frac{a}{u}n, \frac{1}{u}n]$, for any $u > 1$.

1.1 Lévy(μ), Lévy [0,1]

Let μ be a nonnegative measure on $(0, \infty)$, such that $\int x\mu(dx) \in (0, \infty)$. We say that the distribution of T is **Lévy**(μ) if

$$C(\theta) := \log \mathbb{E} e^{\theta T} = \int (e^{\theta x} - 1)\mu(dx), \quad (8)$$

that is, if T has the infinitely divisible distribution with Lévy measure μ . Informally, T is the sum of all arrivals, in the Poisson process on $(0, \infty)$ with intensity $\mu(dx)$. (There are other infinitely divisible distributions Lévy(μ) with Lévy measure supported by $(0, \infty)$, having infinite mean; by restricting to the finite mean case, we gain both a simplified form for the Lévy measure, and the use of moment generating functions, rather than needing characteristic functions, to specify the distribution.)

We are mainly interested in the case when the support of μ is bounded. Without loss of generality, by rescaling, the support of μ is contained in $[0, 1]$, and in that case, we say that the distribution of T is in the set Lévy [0,1]. Obviously, if the support of μ is contained in a bounded interval, then $\mathbb{E} e^{\theta T} < \infty$ for all θ , and C is defined everywhere.

1.2 Regularity conditions

The first regularity condition we impose is that μ have a density with respect to Lebesgue measure, say $\mu(dx) = g(x) dx$, so that the distribution of T is determined by

$$C(\theta) := \log \mathbb{E} e^{\theta T} = \int_0^1 (e^{\theta x} - 1)g(x) dx. \quad (9)$$

In this context, the requirement $\int x\mu(dx) \in (0, \infty)$ reduces to $\int_0^1 x g(x) dx \in (0, \infty)$. Other regularity conditions are imposed, as needed, for the proofs of Theorems 1 and 2 stated below.

There are two qualitatively distinct cases, according to whether $\mu((0, 1])$ is finite or infinite. In the first case, $\lambda := \int_0^1 g(x) dx < \infty$, so the total number of arrivals is Poisson distributed with parameter λ , $\mathbb{P}(T = 0) = e^{-\lambda} > 0$, and the distribution of T has a *defective* density f , with $\int f(x) dx = 1 - e^{-\lambda} < 1$.

In the second case, $\int_0^1 g(x) dx = \infty$, and it is not hard to show that the distribution of T has a *proper* density f , with $\int f(x) dx = 1$. An example of the first case is given by (7), with $g(x) = 1/x$ on $(a, 1]$ and $a \in (0, 1)$, and $g(x) = 0$ on $[0, a]$; and an example of this second case, relating to the Dickman function, is the density $f = e^{-\gamma} \rho$ in (6), with $g(x) = 1/x$ on $(0, 1]$.

The two cases need different additional regularity assumptions. For the first case, with a finite number of Poisson arrivals, our main result is given as Theorem 1, which approximates the density $f(u)$ with an $O(1/u)$ upper bound on the relative error, as in the de Bruijn–Alladi result, and approximates the upper tail probability $P(T \geq u)$ with a $O(1/\sqrt{u})$ upper bound on the relative error. The proof, in Section 5, of Theorem 1 relies on a small amount of Fourier analysis, together with the result of Proposition 1, from Section 4, which approximates a tilted density $f_\beta(t)$ with a uniform bound on the *additive* error. The proof of Proposition 1 is the most difficult part of this paper, requiring estimates for *four* different zones of integration.

For the second case, with an infinite number of arrivals, arguments requiring no further Fourier analysis will be given, in Section 6, letting us derive Theorem 2 which gives a $O(1/\sqrt{u})$ upper bound on the relative error for both the density and the upper tail probability.

1.3 Statement of main theorems

For any integer $k \geq 1$, we say that a function $g(x)$ is **piecewise C^k** on an interval $[a, 1]$, if $[a, 1]$ is partitioned into finitely many subintervals on whose interiors g is C^k , and g and all k derivatives all possess finite one-sided limits at all the endpoints. So in particular, g is bounded on $[a, 1]$. We will usually focus on piecewise C^2 .

Theorem 1. *Assume that the non-negative function $g(x)$ defined on $[0, 1]$ is piecewise C^2 on $[0, 1]$, and that for some $\epsilon > 0$, we have*

$$g(x) \geq \epsilon \text{ on } [1 - \epsilon, 1]. \quad (10)$$

Let the distribution of T be given by (9) and let f be the defective density function for T . Given $u > 0$, let $\beta = \beta(u)$ be such that $C'(\beta) = u$; let

$\sigma_\beta^2 = C''(\beta)$. Then as $u \rightarrow \infty$

$$f(u) = \frac{e^{C(\beta)-u\beta}}{\sqrt{2\pi} \sigma_\beta} \left(1 + O\left(\frac{1}{u}\right) \right) \quad (11)$$

and

$$P(T \geq u) = \frac{1}{\beta} \frac{e^{C(\beta)-u\beta}}{\sqrt{2\pi} \sigma_\beta} \left(1 + O\left(\frac{1}{\sqrt{u}}\right) \right) \quad (12)$$

and hence $P(T \geq u) = (f(u)/\beta) \left(1 + O\left(\frac{1}{\sqrt{u}}\right) \right)$.

Theorem 2. Fix $a \in (0, 1/4]$. Let the density function $g : (0, 1] \rightarrow [0, \infty)$ satisfy: the restriction of g to $[a, 1]$ is piecewise- C^2 , $\sup_{0 < x \leq 1} xg(x) < \infty$, and $g(x) \geq \epsilon$ on $[1 - \epsilon, 1]$ for some $\epsilon > 0$. Given $u > 0$, let $\beta = \beta(u)$ be such that $C'(\beta) = u$; let $\sigma_\beta^2 = C''(\beta)$. Let f be the (possibly defective) density function for T . Then as $u \rightarrow \infty$

$$f(u) = \frac{e^{C(\beta)-u\beta}}{\sqrt{2\pi} \sigma_\beta} \left(1 + O\left(\frac{1}{\sqrt{u}}\right) \right) \quad (13)$$

and

$$P(T \geq u) = \frac{1}{\beta} \frac{e^{C(\beta)-u\beta}}{\sqrt{2\pi} \sigma_\beta} \left(1 + O\left(\frac{1}{\sqrt{u}}\right) \right) \quad (14)$$

and hence $P(T \geq u) = f(u)/\beta \left(1 + O\left(\frac{1}{\sqrt{u}}\right) \right)$.

Remarks. 1) The density g can, more generally, be assumed to have bounded support anywhere in the non-negative reals; it is only a useful simplifying normalization made here to place the support in $[0, 1]$ with the upper limit of $\text{supp}(g)$ equal, in fact, to 1. So g possesses a discontinuity at $x = 1$, if nowhere else.

2) Some condition on the possible growth of g at $0+$ is necessary for the validity of a strong error term in a local limit result such as (21) in the underlying Proposition 1; but the natural candidate, $\int_0^1 xg(x) dx < \infty$, satisfied by $g(x) = 1/x$ in the Dickman case, does not of itself suffice. (For example, fix $0 < \theta < 1$ and let $g(x)dx = \theta \frac{dx}{x}$ on $(0, 1]$. The density f of the corresponding T satisfies, for all $0 < x \leq 1$, $f(x) = x^{\theta-1}f(1)$, as follows immediately from scale invariance, together with the fact that $P(\text{no arrivals in } (x, 1]) = \exp(-\int_x^1 g(x)dx) = x^\theta$. Thus f is unbounded, and the same

is true after any tilt and standardization. Hence the uniform error estimate in (21) must fail.)

3) The boundedness away from 0 at the rightmost boundary of $\text{supp}(g)$ ensures that the tilted measure $e^{\beta x}g(x)dx$ will have unbounded mass near 1 as the tilting parameter β increases. However, we believe that our proof could be extended to cover the more general case not assuming (10).

4) We conjecture the *conclusions* of Proposition 1 itself remain valid under a far weaker set of hypotheses than any considered here. In fact, we believe it suffices to assume the intensity measure ν of the Poisson process $\text{PP}(\nu)$ has a non-trivial absolutely continuous part, and the sum T of arrivals have finite mean value; see Section 7 for an explicit statement.

2 An easy bulk CLT

Although the result proved in this section, Theorem 3, is an easy exercise, it serves well to introduce our notation for Cramér tilts, and to show how our local limit results are much more delicate than the bulk central limit theorem; see also Conjecture 1 in Section 7. We give an example, after Theorem 3, to show that the conclusion of the theorem can fail without the hypothesis that the support of μ is bounded.

For $\beta \in (-\infty, \infty)$, let T_β be distributed as the Cramér β -tilt of T , that is, $\mathbb{P}(T_\beta \in dx) = \mathbb{P}(T \in dx) e^{\beta x} / \mathbb{E} e^{\beta T}$, so that with C_β defined by $C_\beta(\cdot) := C(\cdot + \beta) - C(\beta)$, the distribution of T_β has cumulant generating function C_β . Thus, for our T as given by (8),

$$\mathbb{E} e^{\theta T_\beta} = \exp(C_\beta(\theta)) = \exp\left(\int (e^{\theta x} - 1) e^{\beta x} \mu(dx)\right). \quad (15)$$

Informally, T_β is the sum of all arrivals, in the Poisson process on $(0, 1]$ with intensity $e^{\beta x} \mu(dx)$.

Theorem 3. *Let T be distributed as per (8), with μ supported by $(0, 1]$ and $\int x \mu(dx) \in (0, \infty)$. Consider the Cramér tilts of T , as given by (15). As β grows so that $\mathbb{E} T_\beta \rightarrow \infty$, $(T_\beta - \mathbb{E} T_\beta) / \sqrt{\text{Var} T_\beta}$ converges in distribution to the standard normal.*

Proof. The mean and variance of T_β , call them $u(\beta)$ and $\sigma^2(\beta)$, are given by $u(\beta) = C'_\beta(0) = \int x e^{\beta x} \mu(dx)$ and $\sigma^2(\beta) = C''_\beta(0) = \int x^2 e^{\beta x} \mu(dx)$. Clearly,

$\infty > u(\beta) \geq \sigma^2(\beta) \rightarrow \infty$ as $\beta \rightarrow \infty$, using the hypotheses that $\int x\mu(dx) \in (0, \infty)$ and that the support of μ is contained in $[0, 1]$. It is easy to see, from the explicit integrals for $u(\beta)$ and $\sigma^2(\beta)$, that with $x_0 := \sup(\text{support}(\mu)) \in (0, 1]$, $\sigma^2(\beta)/u(\beta) \rightarrow x_0$ as $\beta \rightarrow \infty$. To set up use of the Lindeberg-Feller central limit theorem, fix a sequence $\beta(n) \rightarrow \infty$, and take a triangular array where the n th row has $m(n)$ i.i.d. mean zero entries, for $m(n) = \lceil u(\beta(n)) \rceil$, and the sum of these $m(n)$ entries is $(T_{\beta(n)} - \mathbb{E} T_{\beta(n)})$. In other words, each entry in row n is distributed as $Y_n - \mathbb{E} Y_n$, where

$$\mathbb{E} e^{Y_n} = \exp\left(\left(\int (e^{\theta x} - 1) \frac{1}{m(n)} e^{\beta x} \mu(dx)\right)\right).$$

Note that $m(n) \sim u(\beta(n))$ as $n \rightarrow \infty$. The sum of the variances for the n th row is $\sigma^2(\beta(n))$, with $\sigma^2(\beta(n)) \sim x_0 u(\beta) \sim x_0 m(n)$. The hypothesis of the Lindeberg-Feller theorem, for any triangular array having $m(n)$ independent entries each distributed as Y_n , with total variance $\sigma^2(\beta(n))$ for the n th row, is that for fixed $\varepsilon > 0$,

$$m(n) \mathbb{E}((Y_n - \mathbb{E} Y_n)^2; |Y_n - \mathbb{E} Y_n| > \varepsilon \sigma(\beta(n))) = o(\sigma^2(\beta(n))); \quad (16)$$

hence for our setup we need only show that

$$\mathbb{E}((Y_n - \mathbb{E} Y_n)^2; |Y_n - \mathbb{E} Y_n| > \varepsilon \sigma(\beta(n))) = o(1)$$

as $n \rightarrow \infty$.

For sufficiently large n , $\varepsilon \sigma(\beta(n)) > 1$, and since $Y_n \geq 0$ and $\mathbb{E} Y_n \leq 1$, for these sufficiently large n we have

$$\mathbb{E}((Y_n - \mathbb{E} Y_n)^2; |Y_n - \mathbb{E} Y_n| > \varepsilon \sigma(\beta(n))) = \mathbb{E}((Y_n - \mathbb{E} Y_n)^2; Y_n - \mathbb{E} Y_n > \varepsilon \sigma(\beta(n))),$$

which in turn is at most

$$\mathbb{E}(Y_n^2; Y_n > \varepsilon \sigma(\beta(n))).$$

Finally, [3, Theorem 1.2 and Section 6] assert that *any* random variable X in Lévy[0,1], with $\mathbb{E} X \leq 1$, satisfies, for all $t \geq 1$, $\mathbb{P}(X \geq t) \leq 1/\Gamma(1+t)$. Our Y_n is of this form, so the upper bound on the upper tail probability gives $\mathbb{E}(Y_n^2; Y_n \geq x) = x^2 \mathbb{P}(Y_n \geq x) + \int_x^\infty 2t \mathbb{P}(Y_n \geq t) dt \leq x^2/\Gamma(1+x) + \int_{t \geq x} 2t/\Gamma(1+t) dt = o(1)$, using $x = \varepsilon \sigma(\beta(n)) \rightarrow \infty$.

□

Example Take μ to be the measure on $(0, \infty)$ with $\mu(dx) = e^{-x}/x \, dx$; this is known as the *Moran subordinator*, see for example [10, Section 9.4]. With T and T_β as given by (8) and (15), T has the standard exponential distribution, and for $\beta < 1$, T_β has the exponential distribution with mean $1/(1 - \beta)$. For any $u \in (0, \infty)$, one can solve $\mathbb{E} T_\beta = u$, and as $u \rightarrow \infty$, we have $\beta(u) \rightarrow 1$ but there is no rescaling and centering of T_β which converges to the normal distribution.

3 Preliminaries

In this section we fix notation and prove some preliminary lemmas about the distribution of the sum of arrivals, T , in a Poisson process $\text{PP}(\nu)$ on the non-negative reals, with intensity measure ν satisfying

$$d\nu = g(x)dx$$

for some density function $g(x)$ satisfying certain subsets of the hypotheses of Theorem 1. For omitted proofs or definitions pertaining to Poisson processes, we refer the reader to [10].

For any intensity measure ν with support in $[0, 1]$, define

$$C(z) := \int_0^1 (e^{zx} - 1) d\nu(x), \quad (17)$$

and let T denote the sum of arrivals in $\text{PP}(\nu)$. When $d\nu = e^{\beta x} g(x) dx$ for some fixed $g(x)$, we may write T_β and C_β , but in Lemma 1 below, we suppress dependence on β , to avoid clutter. We sometimes specifically single out the case $\beta = 0$, i.e., the untilted measure, with the subscript “0”. Thus

$$C_0(z) = \int_0^1 (e^{zx} - 1) g(x) dx \quad (18)$$

and

T_0 is the sum of arrivals in $\text{PP}(g(x)dx)$.

Trivially, for $d\nu = e^{\beta x} g(x) dx$, we have

$$C(z) = C_0(z + \beta) - C_0(z). \quad (19)$$

Lemma 1. Let ν and T be as just discussed. Then

- a) $Ee^{zT} = e^{C(z)}$
- b) $ET = \int_0^1 x d\nu(x)$, and
- c) $\text{var}(T) = \int_0^1 x^2 d\nu(x)$.

Proof. These are parts of Campbell's theorem, valid for very general intensity measures ν ; see [10]. \square

Lemma 2. Let $d\nu = e^{\beta x} g(x) dx$, where $g(x)$ is nonnegative and bounded and, for some $\varepsilon > 0$, satisfies (10). Then

- a) $Ee^{zT_\beta} = e^{C_0(z+\beta) - C_0(\beta)}$.
- b) There are constants $0 < K_1 < K_2$ (depending on g), such that $C_0(\beta)$, together with any finite collection of integrals $\int_0^1 x^k e^{\beta x} g(x) dx$ for $k = 0, 1, 2, \dots$ all lie between $K_1 \frac{e^\beta}{\beta}$ and $K_2 \frac{e^\beta}{\beta}$ for β sufficiently large. (The constants also depend on the particular finite collection.) If only $xg(x)$ is bounded, instead of $g(x)$, this still holds for $k = 1, 2, \dots$.
- c) There are (different) constants $0 < K_1 < K_2$ such that for β sufficiently large, the ratio of any pair of integrals from part b) lies between K_1 and K_2 .

Proof. a) is immediate from (19) and c) is immediate from b). As for b), when $g(x)$ is bounded our hypotheses imply that for some $K, \epsilon > 0$, for any non-negative function $h(x)$ we have

$$\epsilon \int_{1-\epsilon}^1 h(x) e^{\beta x} dx \leq \int_0^1 h(x) e^{\beta x} g(x) dx \leq K \int_0^1 h(x) e^{\beta x} dx.$$

The rest is integration by parts.

If only $xg(x)$ is bounded then to get the rightmost inequality, for any non-negative $h(x)$ for which $h(x)/x$ is bounded on $[0, 1]$ (such as $h(x) = x$) rewrite the middle integrand as $h(x) e^{\beta x} g(x) = (h(x)/x) e^{\beta x} (xg(x))$ and proceed from there. \square

Lemma 3. Let $d\nu = e^{\beta x} g(x) dx$, where g is nonnegative and bounded and, for some $\varepsilon > 0$, satisfies (10). Then

- a) $C_0(\beta)$, along with all the integrals $\int_0^1 x^k e^{\beta x} g(x) dx$ for $k = 0, 1, 2, \dots$, grows to ∞ as $u(\beta) \rightarrow \infty$.
- b) The following statements only require $xg(x)$ bounded, not $g(x)$:
For $\beta > 0$, the function $u(\beta) := ET_\beta$ satisfies $\frac{du}{d\beta} = \text{var}(T_\beta) > 0$ and,

so, is monotone increasing and hence invertible. The inverse function $\beta(u)$ satisfies $\frac{e^\beta}{u} \rightarrow \infty$ as $u \rightarrow \infty$ (or $\beta \rightarrow \infty$), but for any $\epsilon > 0$, $\frac{e^\beta}{u^{1+\epsilon}} \rightarrow 0$.

c) $P(T_\beta = 0) = e^{-\int_0^1 e^{\beta x} g(x) dx} = O(e^{-Ku})$ for some $K > 0$, where $u = ET_\beta$.

Proof. This is all just a corollary of Lemma 2. a) follows at once from Lemma 2(b). As for b), the assertions about $u(\beta)$ are immediate.

To confirm the growth properties of e^β , if we had $e^\beta/u < K$ for arbitrarily large u , for some K , then also we would have

$$u = ET_\beta < K_2 \frac{e^\beta}{\beta} \Rightarrow u < \frac{KK_2 u}{\beta} \Rightarrow 1 < \frac{KK_2}{\beta}$$

for arbitrarily large β , which is impossible.

Also, suppose that for some $\epsilon > 0$ we have $\frac{e^\beta}{u^{1+\epsilon}} > K > 0$, for some K , for arbitrarily large u . Pick $0 < \zeta < 1$ such that $(1+\epsilon)(1-\zeta) > 1$. Since asymptotically $\frac{e^\beta}{\beta} > e^{(1-\zeta)\beta}$ and also $u > K_1 \frac{e^\beta}{\beta}$, we would have $u > K_1 e^{(1-\zeta)\beta} > u^{(1-\zeta)(1+\epsilon)} K_1 K^{1-\zeta}$ for arbitrarily large u , which is also impossible.

Finally, the formula for $P(T_\beta = 0)$ is standard Poisson theory, and the bound follows from the definition of u with Lemma 1(b) and 2(b). \square

To conclude, we show that when the density g is piecewise C_k for $k \geq 1$, the random variable T_β does, in fact have a density function f_β , possessing a certain degree of regularity. Strictly speaking we should refer to f_β as a *defective* density, since the distribution of T_β has a positive atom at 0. Namely, under our hypotheses, for $\beta \geq 0$ we have

$$\mu := \nu\{[0, 1]\} = \int_0^1 e^{\beta x} g(x) dx < \infty.$$

So in our Poisson process the probability of no arrivals at all, $P(T_\beta = 0)$, is $e^{-\mu} > 0$.

However, conditional on there being any arrivals at all, T_β does possess a conditional density. In fact, we have the following:

Lemma 4. *Let the density function $g(x)$ be piecewise C^k on $[0, 1]$, for $k \geq 1$.*

- a) *Conditional on $T_\beta > 0$, T_β possesses a (conditional) density h .*
- b) *h is piecewise continuous, and for $t > 1$, $h(t)$ is continuous.*
- c) *For $t > 2$, $h(t)$ is C^1 .*

Proof. Let $q(x)$ be the probability density on $[0, 1]$ proportional to $e^{\beta x}g(x)$.

a) Conditional on exactly k arrivals, for $k > 0$, the density of T_β is the k -fold convolution $q * \cdots * q$. (See [10] for this standard result.) Note that these convolution products are uniformly bounded by $\sup(q)$, since for any pair q_1 and q_2 of probability densities on \mathbb{R} we have

$$\begin{aligned}\sup(q_1 * q_2) &= \sup \left\{ \int_{-\infty}^{\infty} q_1(t-s)q_2(s)ds \right\} \\ &\leq \sup(q_1) \cdot \int_{-\infty}^{\infty} q_2(s)ds = \sup(q_1).\end{aligned}$$

Let p_k be the conditional probability of k arrivals, given that there is at least one. Because the q^{*k} are uniformly bounded, the sum $\sum_{k=1}^{\infty} p_k q^{*k}(x)$ exists and is measurable; and since the conditional probability that $a < T_\beta \leq b$, given at least one arrival, is

$$\sum_{k \geq 1} p_k \int_a^b q^{*k}(x)dx,$$

it follows that $h(x)$ is a conditional density for T_β .

b) It is an exercise to check inductively that for $k \geq 2$, the k -fold convolution of k copies of q is continuous. So by uniform convergence, the function

$$h(x) = \sum_{k \geq 2}^{\infty} p_k q^{*k}$$

is continuous. Since $\text{supp}(q) \subset [0, 1]$ it follows that $h = q + \sum_{k \geq 2}^{\infty} p_k q^{*k}$ is continuous for $t > 1$, and piecewise continuous for $t \leq 1$.

c) It is another exercise¹ to show that since q is piecewise C^1 and q^{*k} is continuous for $k \geq 2$, $q^{*(k+1)} = q * (q^{*k})$ is C^1 . We claim, further, that

$$h_3 = \sum_{k=3}^{\infty} p_k q^{*k}$$

is C^1 . If so, then since $\text{supp}(p_1 q + p_2 q * q) \subset [0, 2]$, it follows that h is C^1 for $t > 2$.

¹see, e.g. exercise 2.37 on page 128 of [9]

We show that h_3 is C^1 . Since $\frac{d}{dt}(q * q^{*k}) = q' * q^{*k}$, it follows by the argument in part a) that since q' is bounded, the derivatives of $q^{*(k+1)}$ are uniformly bounded in k . Let $d(t)$ be the series formed from term by term derivatives. Then

$$\begin{aligned} & |(h_3(t + \delta) - h_3(t)) / \delta - d(t)| \\ & \leq \left| \sum_{k=3}^{k_0} p_\beta \left((q^{*k})'(t_k) - (q^{*k})'(t) \right) \right| + \left| \sum_{k \geq k_0} \left((q^{*k})'(t_k) - (q^{*k})'(t) \right) \right| \end{aligned}$$

where $t \leq t_k \leq t + \delta$, by the mean value theorem. Because of all the boundedness and the convergence of $\sum p_k$, the tail can be made arbitrarily small, independent of δ , with sufficiently large K_0 , and then the initial segment can also be made arbitrarily small with sufficiently small δ . So h_3 is C^1 with $h'_3(t) = d(t)$. \square

Writing $p_0 = P(T_\beta = 0)$, the probability measure underlying the distribution of T_β can be written as

$$p_0 \delta_0(t) + (1 - p_0) h(t) dt.$$

The term $(1 - p_0)h$, with total mass $1 - p_0 < 1$, is referred to as a defective density.

4 Statement and proof of Proposition 1

Let $g(x)$ be a bounded density function satisfying the hypotheses of Theorem 1. Let T_β be as given by (15). Let

$$Y \equiv Y_\beta := \frac{T_\beta - E_\beta}{\sigma_\beta} \quad (20)$$

be the standardized version of T_β . Since T_β possesses an atom at 0, Y possesses an atom at $\frac{-E_\beta}{\sigma_\beta}$. By Lemma 4 we know that Y possesses a defective density function $f_Y(y)$. We will prove the following result:

Proposition 1. *If $\beta \rightarrow \infty$, then $E_\beta \rightarrow \infty$ and*

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} + O\left(\frac{1}{\sqrt{E_\beta}}\right) \quad (21)$$

uniformly in y . Further, for $y = 0$ we have the stronger statement

$$f_Y(0) = \frac{1}{\sqrt{2\pi}} + O\left(\frac{1}{E_\beta}\right). \quad (22)$$

Proof. This proof is inspired by the proof of Theorem 2.1 in [8], though we cannot use formulas involving $\hat{\rho}(s)$, and associated quantities, that play a significant role in their treatment.

Write

$$u = ET_\beta, \quad \sigma_\beta^2 = \text{var } T_\beta, \quad \alpha = 1/\sigma_\beta. \quad (23)$$

Since T_β has an atom at 0, $Y = \alpha(T_\beta - u)$ inherits an atom at $-\alpha u$, with probability mass

$$P(T_\beta = 0) = e^{-\int_0^1 e^{\beta x} g(x) dx}.$$

Therefore we can calculate the characteristic function of Y as

$$\varphi(t) := Ee^{itY} = P(T_\beta = 0)e^{-it\alpha u} + \int_{-\infty}^{\infty} e^{ity} f_Y(y) dy.$$

But since by Lemma 2(a) we also have

$$\begin{aligned} \varphi(t) &= Ee^{it(\alpha T_\beta - \alpha u)} \\ &= e^{C_0(\beta + i\alpha t) - C_0(\beta) - i\alpha t u} \end{aligned}$$

we find that the Fourier transform of f_Y takes the form

$$\hat{f}_Y(t) = \int_{-\infty}^{\infty} e^{-ity} f_Y(y) dy = e^{C_0(\beta - i\alpha t) - C_0(\beta) + i\alpha t u} - P(T_\beta = 0)e^{i\alpha t u}.$$

Since f_Y possesses discontinuities in the interval $[-\alpha u, 1 - \alpha u]$, \hat{f}_Y cannot be an L^1 function. Nevertheless, we have

$$\begin{aligned} f_Y(y) &= \lim_{A \rightarrow \infty} \frac{1}{2\pi} \int_{-A}^A e^{iyt} \hat{f}_Y(t) dt \\ &= \lim_{A \rightarrow \infty} \frac{1}{2\pi} \int_{-A}^A e^{iyt} \{e^{C_0(\beta - i\alpha t) - C_0(\beta) + i\alpha t u} - P(T_\beta = 0)e^{i\alpha t u}\} dt, \end{aligned} \quad (24)$$

valid wherever f_Y is differentiable. But from Lemma 4 and Formulas (43) and (44), we know that for any y , $f'_Y(y)$ exists for β sufficiently large, and this

is all we will need. (See [12, Chapter 6, Section 5], particularly Theorem 5.13 on page 243, for the applicable inversion formula.)

We now proceed to evaluate the right hand side of (24), for finite β . We partition the domain of integration, in (24), into four concentric zones around $t = 0$ and work on them separately. Choose and fix $r > 3\pi^2/4$. The zones are

$$\begin{aligned} \text{Zone 0: } & -R(u) \leq t \leq R(u), \text{ where } R(u) = \sqrt{r \log u}. \\ \text{Zone 1: } & [-\pi\sigma_\beta, -R(u)] \cup [R(u), \pi\sigma_\beta] \\ \text{Zone 2: } & [-\beta\sigma_\beta^2, -\pi\sigma_\beta] \cup [\pi\sigma_\beta, \beta\sigma_\beta^2] \\ \text{Zone 3: } & [-A, -\beta\sigma_\beta^2] \cup [\beta\sigma_\beta^2, A] \text{ (where } A \rightarrow \infty). \end{aligned}$$

Proposition 2. *The contributions to (24) from the above four zones, are, respectively,*

$$\begin{aligned} \text{Zone 0: } & \frac{1}{\sqrt{2\pi}} e^{-y^2/2} + O\left(\frac{1}{\sqrt{u}}\right) \quad \text{if } y \neq 0; \\ & \frac{1}{\sqrt{2\pi}} + O\left(\frac{1}{u}\right) \quad \text{if } y = 0. \\ \text{Zone 1: } & o\left(\frac{1}{u}\right). \\ \text{Zone 2: } & o\left(\frac{1}{u^k}\right) \quad \text{for any } k > 0. \\ \text{Zone 3: } & o\left(\frac{1}{u^k}\right) \quad \text{for any } k > 0. \end{aligned}$$

Because the integral is the sum of the four contributions, this will prove Proposition 1.

Proof of Proposition 2. For the first three zones we can neglect the term

$$P(T_\beta = 0) e^{i\alpha t u}$$

in the integrand of (24). In fact, the following is true:

Lemma 5. *We have*

$$P(T_\beta = 0) \int_{-\beta\sigma_\beta^2}^{\beta\sigma_\beta^2} e^{i(y+\alpha u)t} dt = O(e^{-Ku}) \text{ for some } K > 0.$$

Proof. The integral is absolutely bounded by $2\beta\sigma_\beta^2$, and by Lemmas 2 and 3

$$P(T_\beta = 0)\beta\sigma_\beta^2 \leq K'e^{-K'u}u^2 \log u$$

for some $K' > 0$; so let $K = K'/2$. \square

Remark: We do not actually need the boundedness assumption on $g(x)$ in the analysis of the first three zones—just the finiteness of ET_0 will do. We most definitely use the boundedness in Zone 3, however, and at present see no way to dispense with that requirement, or something close to it.

Zone 0: We will show that modulo the neglected $O(e^{-ku})$ term we have

$$\frac{1}{2\pi} \int_{-R(u)}^{R(u)} e^{iyt} \hat{f}_Y(t) dt = \frac{1}{2\pi} \int_{-R(u)}^{R(u)} e^{iyt} [e^{C_0(\beta-i\alpha t)-C_0(\beta)+i\alpha tu}] dt. \quad (25)$$

Since $u = \int_0^1 x e^{\beta x} g(x) dx$, by Taylor's theorem we may write

$$C_0(\beta - i\alpha t) - C_0(\beta) + i\alpha tu = \int_0^1 \{e^{(\beta-i\alpha t)x} - e^{\beta x} + i\alpha t x e^{\beta x}\} g(x) dx \quad (26)$$

$$= \int_0^1 e^{\beta x} \left\{ \frac{-\alpha^2 t^2 x^2}{2} + \frac{i^3 \alpha 3 t^3 x^3}{3!} + \alpha^4 t^4 x^4 \cdot O(1) \right\} g(x) dx \quad (27)$$

as $\alpha t x \rightarrow 0$. Since $\alpha R(u) \leq K \sqrt{\frac{\log u}{u}}$ for some $K > 0$, and $|x| \leq 1$, we do have $\alpha t x \rightarrow 0$ as $\beta \rightarrow \infty$, uniformly in t within Zone 0. So by Lemma 2, (26) becomes

$$-\frac{t^2}{2} + \alpha t^3 O_3(1) + \alpha^2 t^4 O_4(1) \quad (28)$$

where $O_3(1)$ is independent of t , not just uniform. Then Taylor's theorem applied again tells us

$$e^{C_0(\beta-i\alpha t)-C_0(\beta)+i\alpha tu} = e^{-t^2/2} (1 + \alpha t^3 O_3(1) + \alpha^2 t^4 O_4(1)) + O\left([\alpha t^3 + \alpha^2 t^4]^2\right). \quad (29)$$

Since $\alpha R^3(u) \leq K \sqrt{\frac{\log^3 u}{u}} \rightarrow 0$ as $u \rightarrow \infty$ and $\alpha^2 R^4(u) \leq K \frac{\log^2 u}{u} \rightarrow 0$ as $u \rightarrow \infty$, the remainder terms will die.

Inserting the terms of (29) into (25), one by one, we find, first

$$\frac{1}{2\pi} \int_{-R(u)}^{R(u)} e^{iyt} e^{-t^2/2} dt = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} + O\left(\frac{1}{R(u)} e^{-R^2(u)/2}\right) \quad (30)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-y^2/2} + O\left(\frac{1}{\sqrt{\log u}} \frac{1}{u^{r/2}}\right);$$

since $r > 2$, the remainder is subdominant. (See the remarks in the analysis of Zone 1 concerning the size of r .)

Next, if $y = 0$, then $\frac{1}{\sqrt{2\pi}} \int_{-R(u)}^{R(u)} e^{iyt} e^{-t^2/2} \alpha t^3 dt \cdot O_3(1) = 0$ by symmetry. Otherwise, since all moments of $e^{-t^2/2} dt$ are finite, this term contributes $\alpha O(1) = O\left(\frac{1}{\sqrt{u}}\right)$. Similarly, the t^4 term, as well as all subsequent terms, can contribute at most $O(\alpha^2) = O\left(\frac{1}{u}\right)$.

This completes the analysis in Zone 0. For Zones 1 and 2, we need a lemma adapted from the analysis of inequalities (2.20) in [8], in which our $C_0(\beta)$ plays the role of their $E(\xi)$.

Lemma 6. *For τ real, let*

$$H(\tau) = C_0(\beta) - \operatorname{Re} \{C_0(\beta - i\tau)\}.$$

Then

- a) *For $|\tau| \leq \pi$, $H(\tau) \geq \frac{2\tau^2}{\pi^2} \sigma_\beta^2$*
- b) *For $|\tau| \geq \pi$ and β sufficiently large,*

$$H(\tau) > \frac{\epsilon \pi^2}{8} \frac{e^\beta}{\beta^3},$$

where ϵ is the lower bound for $g(x)$ near $x = 1$.

Proof. a)

$$\begin{aligned} H(\tau) &= \int_0^1 (e^{\beta x} - 1) g(x) dx - \int_0^1 (e^\beta \cos \tau x - 1) g(x) dx \\ &= \int_0^1 e^{\beta x} (1 - \cos \tau x) g(x) dx \\ &\geq \int_0^1 e^{\beta x} \frac{2\tau^2 x^2}{\pi^2} g(x) dx \\ &= \frac{2\tau^2}{\pi^2} \sigma_\beta^2. \end{aligned}$$

b)

$$\begin{aligned}
H(\tau) &= \int_0^1 e^{\beta x} (1 - \cos \tau x) g(x) dx \\
&\geq \epsilon \int_{1-\epsilon}^1 e^{\beta x} (1 - \cos \tau x) dx \\
&= \operatorname{Re} \left\{ \epsilon \int_{1-\epsilon}^1 e^{\beta x} (1 - e^{i\tau x}) dx \right\} \\
&= \frac{\epsilon}{\beta} (e^\beta - e^{(1-\epsilon)\beta}) - \operatorname{Re} \left\{ \left[\frac{\epsilon e^{i\theta}}{|\beta + i\tau|} \right] [e^{\beta+i\tau} - e^{(1-\epsilon)(\beta+i\tau)}] \right\} \\
&\quad \left(\text{where } \theta = \arg \frac{1}{\beta + i\tau} \right) \\
&\geq \frac{\epsilon}{\beta} (e^\beta - e^{(1-\epsilon)\beta}) - \epsilon \frac{e^\beta + e^{(1-\epsilon)\beta}}{|\beta + i\tau|} \\
&= \frac{\epsilon e^\beta}{\beta} \left[1 - e^{-\epsilon\beta} - \frac{1 + e^{-\epsilon\beta}}{\sqrt{1 + \pi^2/\beta^2}} \right] \\
&\geq \frac{\epsilon e^\beta}{\beta} \left[1 - e^{-\epsilon\beta} - (1 + e^{-\epsilon\beta}) \left(1 - \frac{\pi^2}{4\beta^2} \right) \right] \\
&= \frac{\epsilon e^\beta}{\beta} \left[\frac{\pi^2}{4\beta^2} - \left(2 - \frac{\pi^2}{4\beta^2} \right) e^{-\epsilon\beta} \right] \\
&> \frac{\epsilon e^\beta}{\beta} \cdot \frac{\pi^2}{8\beta^2}
\end{aligned}$$

for β sufficiently large.

□

Now we do the Zone 1 and Zone 2 estimates. Both involve straightforward estimates.

Zone 1: For the upper half of Zone 1 we write

$$\begin{aligned}
& \left| \int_{R(u)}^{\pi\sigma_\beta} e^{iyt} \{e^{C_0(\beta-i\alpha\tau)-C_0(\beta)+i\alpha\tau u}\} dt \right| \\
& \leq \int_{R(u)}^{\pi\sigma_\beta} e^{-H(\alpha t)} dt \\
& = \sigma_\beta \int_{R(u)/\sigma_\beta}^{\pi} e^{-H(\tau)} d\tau \\
& \leq \sigma_\beta \int_{R(u)/\sigma_\beta}^{\pi} e^{-\frac{2\sigma_\beta^2}{\pi^2}\tau^2} d\tau \\
& \leq \pi\sigma_\beta e^{-2R(u)^2/\pi^2} \\
& = \frac{\pi\sigma_\beta}{u^{2r/\pi^2}} = O\left(\frac{1}{u^{2r/\pi^2-1/2}}\right) = o\left(\frac{1}{u}\right)
\end{aligned}$$

since $r > \frac{3\pi^2}{4}$ and $\sigma_\beta = O(\sqrt{u})$. (If $r > (2k+1)\pi^2/4$, we can even get $o(\frac{1}{u^k})$ for any desired k .) The estimate for the lower half is the same.

Zone 2: For the upper half we write

$$\begin{aligned}
& \left| \int_{\pi\sigma_\beta}^{\beta\sigma_\beta} e^{iyt} \{e^{C_0(\beta-i\alpha\tau)-C_0(\beta)+i\alpha\tau u}\} dt \right| \\
& \leq \int_{\pi\sigma_\beta}^{\beta\sigma_\beta} e^{-H(\alpha t)} dt \\
& = \sigma_\beta \int_{\pi}^{\beta\sigma_\beta} e^{-H(\tau)} d\tau \\
& \leq \sigma_\beta \int_{\pi}^{\beta\sigma_\beta} e^{-\frac{\epsilon\pi^2}{8}\frac{e^\beta}{\beta^3}} d\tau \\
& \leq \beta\sigma_\beta e^{-\frac{\epsilon\pi^2}{8}\frac{e^\beta}{\beta^3}} \quad \text{for } \beta \text{ sufficiently large,} \\
& \leq e^{-u^{1-\delta}} \quad \text{for some } \delta < 1, \text{ by Lemmas 2 and 3,}
\end{aligned}$$

which is $o(\frac{1}{u^k})$ for any $k > 0$.

Since once again the estimate of the lower half is the same, this completes the analysis for Zone 2.

Zone 3: Because $P(T_\beta = 0) = e^{-\int_0^1 e^{\beta x} g(x) dx}$, the integral (24) in the upper half of Zone 3 is

$$P(T_\beta = 0) \lim_{A \rightarrow \infty} \frac{1}{2\pi} \int_{\beta \sigma_\beta^2}^A e^{i(y+\alpha u)t} \left\{ e^{\int_0^1 e^{(\beta-i\alpha t)x} g(x) dx} - 1 \right\} dt. \quad (31)$$

By Lemma 3,

$$P(T_\beta = 0) = O(e^{-Ku}) \quad (32)$$

for some $K > 0$. As for the rest of (31), for fixed β we have

$$\lim_{t \rightarrow \infty} \int_0^1 e^{(\beta-i\alpha t)x} g(x) dx = 0,$$

by the Riemann–Lebesgue lemma. So, setting

$$I = I(\beta, t) = \int_0^1 e^{(\beta-i\alpha t)x} g(x) dx, \quad (33)$$

this encourages us to write

$$e^{I(\beta, t)} - 1 = I(\beta, t) + O(I^2(\beta, t)) \quad (34)$$

and substitute (34) for the braced expression in (31). But we need more detailed information about $I(\beta, t)$ before we can exploit the oscillatory factor $e^{i(y+\alpha u)t}$ in (31).

Lemma 7. *There are finitely many numbers*

$$0 \leq a_0 < a_1 < \dots < a_L = 1$$

and

$$C_0, \dots, C_L,$$

with $C_L \neq 0$, such that

$$I(\beta, t) = \sum_{j=0}^L C_j \frac{e^{a_j(\beta-i\alpha t)}}{\beta - i\alpha t} + O\left(\frac{\sigma_\beta^2 e^{2\beta}}{t^2}\right). \quad (35)$$

Proof. This is just integration by parts. Let

$$0 \leq a_0 < a_1 \cdots < a_L = 1$$

be the discontinuity points of g . The contribution to I from the subinterval $[a_j, a_{j+1}]$ is

$$\begin{aligned} & \int_{a_j}^{a_{j+1}} e^{(\beta - i\alpha t)x} g(x) dx \\ &= \frac{1}{\beta - i\alpha t} (g(a_{j+1})e^{(\beta - i\alpha t)a_{j+1}} - g(a_j)e^{(\beta - i\alpha t)a_j}) - \frac{1}{\beta - i\alpha t} \int_{a_j}^{a_{j+1}} e^{(\beta - i\alpha t)x} g'(x) dx \end{aligned}$$

where $g(a_j)$ and $g(a_{j+1})$ are evaluated as one-sided limits, where necessary, and

$$\frac{1}{(\beta - i\alpha t)} \int_{a_j}^{a_{j+1}} e^{(\beta - i\alpha t)x} g'(x) dx = O\left(\frac{\sigma_\beta^2 e^\beta}{t^2}\right)$$

via another integration by parts. (This is where we use the hypothesis that g is piecewise C^2 !) Now collect all the terms from all the subintervals. This completes the proof of Proposition 1. \square

Having Lemma 7, we can use (35) as follows. For $a \leq 1$ and for any fixed y we have

$$\lim_{A \rightarrow \infty} \int_{\beta\sigma_\beta}^A e^{i(y+\alpha u)t} \frac{e^{a(\beta - i\alpha t)}}{\beta - i\alpha t} dt \quad (36)$$

$$= \lim_{A \rightarrow \infty} \sigma_\beta e^{a\beta} \int_{\beta\sigma_\beta}^A \frac{e^{ict}}{\beta - it} dt \quad (\text{where } c = y/\alpha + u - a) \quad (37)$$

$$= \lim_{A \rightarrow \infty} \sigma_\beta e^{a\beta} \left\{ \frac{e^{ict}}{ic(\beta - it)} \Big|_{\beta\sigma_\beta}^A + \int_{\beta\sigma_\beta}^A \frac{e^{ict}}{c(\beta - it)^2} dt \right\} \quad (38)$$

$$= \sigma_\beta e^{a\beta} O\left(\frac{1}{c(\beta - i\beta\sigma_\beta)}\right) \quad (39)$$

$$= O\left(\frac{\sigma_\beta e^\beta}{\beta u^{3/2}}\right) = O(1) \quad (40)$$

using Lemmas 2 and 3.

Also, looking at the big- O term in (35) gives

$$\lim_{A \rightarrow \infty} \int_{\beta \sigma_\beta^2}^A e^{i(y+\alpha u)t} O\left(\frac{\sigma_\beta^2 e^{2\beta}}{t^2}\right) dt = O\left(\frac{e^{2\beta}}{\beta}\right). \quad (41)$$

Further, a little algebra with (35) shows that

$$I^2(\beta, t) = O\left(\frac{\sigma_\beta^4 e^{4\beta}}{t^2}\right)$$

is a very generous bound. This gives

$$\lim_{A \rightarrow \infty} \int_{\beta \sigma_\beta^2}^A e^{i(y+\alpha u)t} O(I^2(\beta, t)) dt = O\left(\frac{\sigma_\beta^2 e^{4\beta}}{\beta}\right). \quad (42)$$

Obviously, (42) is the dominant bound. The quantity in (36) is so small because it “sees” the oscillation.

Finally, substituting (34) into (31) and using the bounds just calculated yields a bound of $O\left(e^{-Ku} \frac{\sigma_\beta^2 e^{4\beta}}{\beta}\right)$ for Zone 3, which is certainly $o\left(\frac{1}{u^k}\right)$ for any k .

The lower half of the zone is handled in exactly the same way. This completes the proof of Proposition 1. \square

5 Proof of Theorem 1

Proof. Given Proposition 1 the proof of Theorem 1 is relatively straightforward. Let

$$C_g(\beta) = \int_0^1 (e^{\beta x} - 1) g(x) dx$$

and, given $u > 0$, let $\beta = \beta(u)$ be the solution of

$$C'_g(\beta) = \int_0^1 x e^{\beta x} g(x) dx = u.$$

By Lemmas 1, 2(b) and 3(b) when $\beta = \beta(u)$ we then have $u = E_\beta := ET_\beta$, while the variance $\sigma_\beta^2 = \text{var } T_\beta$ satisfies both $\sigma_\beta^2 = 1/\beta'(u)$, and $\sigma_\beta^2 \sim u$. As usual we define σ_β to be the positive square root.

Since $u = E_\beta \rightarrow \infty$ as $\beta \rightarrow \infty$, for the inverse function we also have $\beta(u) \rightarrow \infty$ as $u \rightarrow \infty$. Just as in the Dickman case, discussed in Section 1, we have

$$f_Y(y) = \sigma_\beta f_\beta(\sigma_\beta y + u), \quad (43)$$

and, with $Ee^{\beta T_0} = \exp(\int_0^1 (e^{\beta x} - 1) g(x) dx) = \exp(C_g(\beta))$ we have

$$f_\beta(t) = e^{\beta t} f(t) / \exp(C_g(\beta)). \quad (44)$$

We combine the two equations above, to express the density f of T_0 at the point $t = u + \sigma_\beta y$ which is, for T_β , y standard deviations above its mean, u :

$$f(u + \sigma_\beta y) = e^{C_g(\beta) - u\beta} e^{-\beta \sigma_\beta y} f_\beta(u + \sigma_\beta y) \quad (45)$$

$$= e^{C_g(\beta) - u\beta} e^{-\beta \sigma_\beta y} \frac{1}{\sigma_\beta} f_Y(y). \quad (46)$$

We now complete the proof of (11) by taking $y = 0$ in (46) and then using (22).

To prove (12), we combine (21) error term, i.e., the standard $f_Y(y)$, with (45) and (46), to find

$$\mathbb{P}(T_0 \geq u) = \sigma_\beta \int_0^\infty f(u + \sigma_\beta y) dy \quad (47)$$

$$= \frac{e^{C_g(\beta) - u\beta}}{\sqrt{2\pi}} \int_0^\infty e^{-\beta \sigma_\beta y} \left[e^{-y^2/2} + O(1/\sqrt{u}) \right] dy. \quad (48)$$

The main term of the integral in (48) is $\int_0^\infty e^{-\beta \sigma_\beta y} e^{-y^2/2} dy = 1/(\beta \sigma_\beta)(1 + O(1/u))$; we see this approximation in two steps. The upper bound is simply $\int_0^\infty e^{-\beta \sigma_\beta y} e^{-y^2/2} dy < \int_0^\infty e^{-\beta \sigma_\beta y} dy = 1/(\beta \sigma_\beta)$. A lower bound, for any $0 < d$, is $\int_0^\infty e^{-\beta \sigma_\beta y} e^{-y^2/2} dy \geq e^{-d^2/2} \int_0^d e^{-\beta \sigma_\beta y} dy = 1/(\beta \sigma_\beta) e^{-d^2/2} (1 - e^{-d\beta \sigma_\beta})$. We can use $d = 1/\sigma_\beta$, so that $e^{-d^2/2} = 1 - O(1/\sigma_\beta^2) = 1 - O(1/u)$ and $(1 - e^{-d\beta \sigma_\beta}) = 1 - e^{-\beta} = 1 - o(1/u)$, using Lemma 2(b).

To bound the error term in the integral in (48), use the uniformity of the error term in (21). After applying absolute value and taking the absolute value inside, we have an upper bound of the form, with some fixed $K < \infty$, $\int_0^\infty e^{-\beta \sigma_\beta y} K/\sqrt{u} dy = 1/(\beta \sigma_\beta) \times O(1/\sqrt{u})$. This completes the proof of (12). \square

6 Proof of Theorem 2

We will actually derive Theorem 2 as a corollary of Theorem 1 by means of the following strategy: given a density g , satisfying the hypotheses of Theorem 2, which blows up at $0+$, consider, for some fixed $0 < a < 1$, the Poisson process restricted to $[a, 1]$. Write $T^{\ddagger\ddagger}$ for the sum of arrivals in $[a, 1]$; this is a process to which both Lemma 4 and Proposition 1 apply.

The sum T^\ddagger of arrivals in $(0, a)$ is independent of $T^{\ddagger\ddagger}$; we have $T = T^\ddagger + T^{\ddagger\ddagger}$, and T^\ddagger is relatively small. The distribution of T is given by convolving the distribution of T^\ddagger , over which we have relatively little control, with the distribution of $T^{\ddagger\ddagger}$, which has an atom at 0, and a density well-controlled by the above-mentioned results. What, then, can we say about the density of T ? Since $T^{\ddagger\ddagger}$ has an atom at 0, and the density of T^\ddagger may be unbounded at $0+$, a uniform *everywhere* approximation for the density of T , such as (21) is not possible—it fails at $y = -u/\sigma_\beta$. But to prove our result, we will only need the uniform approximation at $y \geq 0$.

It was clearly necessary to require that $\int_0^1 xg(x) dx < \infty$, so that T is a random variable with finite values; and we have, in fact, imposed a slightly more restrictive hypothesis, namely that $\sup_{0 < x \leq 1} xg(x) < \infty$. This contrasts with Lemma 4, in which we allow $\int_0^1 g(x) dx$ to be either finite or infinite.

Lemma 8. *Suppose that $L := \sup_{0 < x \leq 1} xg(x) < \infty$. Then T^\ddagger has a (possibly defective) density f , with $\sup_{0 < x \leq 1} xf(x) \leq L < \infty$.*

Proof. Since T^\ddagger is a sum of Poisson arrivals, its size-biased distribution is that of a random variable T^* which satisfies

$$T^* = T^\ddagger + I \text{ in distribution,}$$

with T^\ddagger, I independent and I having density $xg(x)/c$, where $c = \int_0^1 xg(x) dx$ —see [4]. Since the distribution of T^* is the convolution of a probability distribution, namely that of T^\ddagger , with an absolutely continuous distribution, namely that of I , the distribution of T^* has a proper probability density, say f^* . We infer from the convolution equation $T^* = T^\ddagger + I$ that even if the distribution of T^\ddagger has an atom at 0, it must also have a (possibly defective) density f on $(0, \infty)$ satisfying $xf(x)/c = f^*(x)$. Then the convolution equation at the density level reduces, after multiplication by a factor of c on both sides, to

$$xf(x) = \int_0^1 f(x-z)zg(z) dz$$

which implies that $\sup xf(x) \leq \sup xg(x)$. \square

We next give an upper bound for use with the “relatively small” contributions from T_β^\dagger .

Lemma 9. *Given $0 < m < \infty$, let ν be any measure on $(0, 1]$ such that $m = \int x d\nu$, and let W be the sum of arrivals in the Poisson process $\text{PP}(\nu)$, so that $EW = m$. Uniformly over choices of ν , the chance that W exceeds twice its mean decays exponentially fast relative to $m \rightarrow \infty$; in fact*

$$P(W \geq 4EW) \leq \exp(-m(4 - e)).$$

Proof. Since ν is supported by $(0, 1]$, $Ee^W = \exp(\int (e^x - 1)d\nu) \leq \exp(\int e x d\nu) = \exp(em)$. Hence

$$P(W \geq 4EW) = P(e^W \geq e^{4m}) \leq Ee^W / e^{4m} \leq e^{em} / e^{4m}.$$

\square

We apply Lemma 9 to the random variables T_β^\dagger to show that the chance of strictly exceeding 4 times the mean is $o(1/u)$, where u is defined by $\beta = \beta(u)$ — if $T^\dagger = 0$ identically, the probability is zero. Otherwise there is some $\epsilon > 0$ with $\int_{2\epsilon}^a g(x) dx > \epsilon$. When we tilt T to get $ET_\beta = u$, by Lemma 3(b), we know that $\beta \sim \log u$, hence for sufficiently large u we have $\beta > .5 \log u$, and hence $ET_\beta^\dagger \geq \epsilon^2 u^\epsilon$. The upper bound from Lemma 9 then gives

$$P(T_\beta^\dagger > 4ET_\beta^\dagger) \leq \exp(-(4 - e)\epsilon^2 u^\epsilon) = o(1/u). \quad (49)$$

We can now prove Theorem 2:

Proof of Theorem 2. As outlined in the second paragraph of this section, write $T = T^\dagger + T^{\dagger\dagger}$ for the sum of arrivals in the Poisson process with arrival intensity $g(x)dx$. Write $T_\beta = T_\beta^\dagger + T_\beta^{\dagger\dagger}$ for the same, after tilting.

We apply Proposition 1 to $T^{\dagger\dagger}$, using the β for which $u = ET_\beta$. We must be careful to note that the mean and variance used to standardize $T_\beta^{\dagger\dagger}$ are *not* the mean u and variance σ_β^2 of T_β . To emphasize this, we write

$$u^{\dagger\dagger} := ET_\beta^{\dagger\dagger} = \int_a^1 x e^{\beta x} g(x) dx, \quad \sigma_\beta^{\dagger\dagger 2} = \int_a^1 x^2 e^{\beta x} g(x) dx,$$

so that (20), specifying the standardized version of $T_\beta^{\dagger\dagger}$, is

$$Y = \frac{T_\beta^{\dagger\dagger} - u^{\dagger\dagger}}{\sigma_\beta^{\dagger\dagger}}. \quad (50)$$

If we write

$$u^\dagger := ET_\beta^\dagger = \int_0^a x e^{\beta x} g(x) dx, \quad \sigma_\beta^{\dagger 2} = \int_0^a x^2 e^{\beta x} g(x) dx,$$

so that $u = u^\dagger + u^{\dagger\dagger}$ and $\sigma_\beta^2 = \sigma_\beta^{\dagger 2} + \sigma_\beta^{\dagger\dagger 2}$ then we see that

$$u^{\dagger\dagger} = u - u^\dagger = u - O(u^a) = u (1 + O(u^{a-1})) \quad (51)$$

and

$$\sigma_\beta^{\dagger\dagger 2} = \sigma_\beta^2 - \sigma_\beta^{\dagger 2} = \sigma_\beta^2 - O(u^a) \sim u (1 + O(u^{a-1})),$$

so

$$\sigma_\beta^{\dagger\dagger} \sim \sqrt{u} (1 + O(u^{a-1})). \quad (52)$$

We write $f_\beta^{\dagger\dagger}$ for the (defective) density of $T_\beta^{\dagger\dagger}$, and f_Y for the (defective) density for the Y in (50). The ordinary change of variables, together with Proposition 1 give, uniformly in y ,

$$f_\beta^{\dagger\dagger}(x) = \frac{1}{\sigma_\beta^{\dagger\dagger}} f_Y\left(\frac{x - u^{\dagger\dagger}}{\sigma_\beta^{\dagger\dagger}}\right) \text{ and } f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} + O\left(\frac{1}{\sqrt{u^{\dagger\dagger}}}\right). \quad (53)$$

Since $T_\beta = T_\beta^\dagger + T_\beta^{\dagger\dagger}$ with independent non-negative summands, we have (see for example [5, page 356])

$$f_\beta(t) = P(T_\beta^{\dagger\dagger} = 0) f_\beta^\dagger(t) + \int f_\beta^{\dagger\dagger}(t - z) dF_\beta^\dagger(z). \quad (54)$$

[We took the integral $dF_\beta^\dagger(z)$ rather than with $f_\beta^\dagger(z) dz$ since T_β^\dagger may have an atom at 0.] By Lemma 8, $\sup t f(t) \leq L$, so for $t > 1$, $f(t) \leq L$ and $f_\beta(t) \leq L e^{\beta t}$. The size of the atom at 0 for $T_\beta^{\dagger\dagger}$ is $\exp(-\int_a^1 e^{\beta x} g(x) dx) \approx \exp(-u)$, so uniformly in $1 \leq t \leq u$ we have $P(T_\beta^{\dagger\dagger} = 0) f_\beta^\dagger(t) = o(1/u)$, and this contribution to (54) is negligible.

To complete the proof of (13) and (14), arguing as in the two paragraphs following (48), we will only need approximations for $f_\beta(t)$ with $t \geq u$, and

especially with $t \in [u, u + 1]$. Also, we will use Lemma 9 to bound the contribution from $dF_\beta^\dagger(z)$ with $z \geq 4u^\dagger$. Hence, we consider $t \in [u, u + 1]$ and $z \in [0, 4u^\dagger]$ so that $x := t - z \in [u - 4u^\dagger, u + 1]$, hence

$$x - u^{\dagger\dagger} \in [u - u^{\dagger\dagger} - 4u^\dagger, u - u^{\dagger\dagger} + 1] = [-3u^\dagger, u^\dagger + 1].$$

Combining these bounds with (51) and (52), uniformly over $t \in [u, u + 1]$ and $z \in [0, 4u^\dagger]$, the argument $y := (x - u^{\dagger\dagger})/\sigma_\beta^{\dagger\dagger}$ to f_Y in (53) is $O(u^{a-.5})$. Hence $y^2 = O(u^{2a-1})$, which is $O(1/\sqrt{u})$ using $a \leq .25$. Combining with the error term written as $O(1/\sqrt{u^{\dagger\dagger}})$, and using $u^{\dagger\dagger} \sim u$, we have, uniformly over $t \in [u, u + 1]$ and $z \in [0, 4u^\dagger]$,

$$f_\beta^{\dagger\dagger}(t - z) = \frac{1}{\sqrt{2\pi} \sigma_\beta^{\dagger\dagger}}(1 + O(1/\sqrt{u})) = \frac{1}{\sqrt{2\pi} \sigma_\beta}(1 + O(1/\sqrt{u})). \quad (55)$$

The second equality in (55), switching from $\sigma_\beta^{\dagger\dagger}$ to σ_β in the denominator, is justified by (52), together with $a \leq .5$, to combine two error terms into one.

The contribution to (54) from integrating over $z > 4u^\dagger$ is $o(1/u)$, using (49) and the fact that $f^{\dagger\dagger}$ is bounded by $\sup_{a \leq x \leq 1} g(x)$. Finally, the contribution to (54) from integrating over $z \in [0, 4u^\dagger]$ is $\frac{1}{\sqrt{2\pi} \sigma_\beta^{\dagger\dagger}}(1 + O(1/\sqrt{u}))$, using (55), and using (49) in the form $\mathbb{P}(T_\beta^\dagger \in [0, 4u^\dagger]) = 1 - O(1/u)$. Thus we have proved that uniformly over $t \in [u, u + 1]$

$$f_\beta(t) = \frac{1}{\sqrt{2\pi} \sigma_\beta}(1 + O(1/\sqrt{u})). \quad (56)$$

A similar argument, now for $t \geq u$ instead of for $t \in [u, u + 1]$, shows that uniformly over $t \geq u$, $\sigma_\beta f_\beta(t) \leq 1 + O(1/\sqrt{u})$.

We apply (45), and argue as we did following (46), to complete the proof of (13) and (14). \square

7 A Concluding Conjecture

In our results we have assumed that the measure μ had a density $g(x)$ satisfying certain regularity conditions. In fact we conjecture that this result can be considerably weakened.

Conjecture 1. *Let μ be any measure on $(0, 1]$ such that $\int x d\mu < \infty$. Allow μ to have atoms and a continuous singular part, but require that the absolutely continuous part of μ , say $g(x)dx$, be nontrivial, in the sense that for some $0 \leq a < b \leq 1$ and $\varepsilon > 0$ we have $g(x) \geq \varepsilon$ for all $x \in (a, b)$. Then the tilted sum of arrivals, T_β , as given by (15) has standardized version Y with density satisfying*

$$f_Y(y) \rightarrow \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \quad (57)$$

for all y , similar to (22) in Proposition 1, but we make no assertion about a rate of convergence. Hence T as given by (8), has (possibly defective) density f satisfying

$$f(u) = \sqrt{\frac{\beta'(u)}{2\pi}} e^{C(\beta) - u\beta} (1 + o(1)) \quad (58)$$

where

$$C(\beta) = \int_0^1 (e^{\beta x} - 1) \mu(dx),$$

similar to (11) in Theorem 1, again with no assertion concerning a rate of convergence.

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